# On the Second Homology of Cactus Groups

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**Abstract**

We use Hopf's formula to show that the second integral homology of cactus groups are elementary abelian 2-groups of finite rank. Consequently, the second rational homology of cactus groups are trivial.

## 1 Introduction

The cactus groups $J_n\,(n \geq 2)$ first appeared in [7] and [6], where they were referred to as *quasibraid groups* and *mock reflection groups*, respectively. The symmetric group $S_n$ on $n$ letters acts on $\overline{M}_0^{n+1}(\mathbb{R})$, the Deligne–Knudson–Mumford moduli space of stable real curves of genus 0 with $n+1$ marked points, by permuting the first $n$ marked points. The cactus group $J_n$ can be identified with the orbifold fundamental group of the quotient orbifold $[\overline{M}_0^{n+1}(\mathbb{R})/S_n]$. Henriques and Kamnitzer [9] proposed the term *cactus group*, inspired by the Opuntia-cactus-like form of stable real curves in the moduli spaces above. Cactus groups also arise in the study of hives and octahedron recurrence [11, 10]. Their generalizations to other Coxeter types have since made them a recurrent tool in representation theory [2, 12, 5].

Concretely, for a given integer $n \geq 2$, the *cactus group of degree* $n$, denoted by $J_n$, defined by the generators $\sigma_{p,q}$ with $1 \leq p < q \leq n$ and the following three relations:

(1) $\sigma_{p,q}^2 = 1 \quad \text{for} \quad 1 \leq p < q \leq n,$

(2) $\sigma_{p,q}\sigma_{r,s} = \sigma_{r,s}\sigma_{p,q} \quad \text{for} \quad [p,q] \cap [r,s] = \emptyset,$

(3) $\sigma_{p,q}\sigma_{r,s} = \sigma_{p+q-s,p+q-r}\sigma_{p,q} \quad \text{for} \quad [r,s] \subset [p,q].$

Here, $[p,q]$ denotes the set $\{p, p+1, \ldots, q-1, q\}$ for positive integers $p, q$ with $p < q$.

If the third relation is omitted, the resulting presentation defines a right-angled Coxeter group $W_n$. Therefore, there is a natural surjection of $W_n$ onto $J_n$. Moreover, there is also a natural surjective homomorphism

$$\pi \colon J_n \to S_n, \quad \sigma_{p,q} \mapsto s_{p,q},$$

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where $s_{p,q} \in S_n$ is the permutation defined by

$$s_{p,q}(i) = \begin{cases} p+q-i, & \text{if } i \in [p,q], \\ i, & \text{otherwise.} \end{cases}$$

The kernel of $\pi$ is called the *pure cactus group of degree* $n$, and is denoted by $PJ_n$.

Since Davis–Januszkiewicz–Scott proved in [6] that the moduli space $\overline{M}_0^{n+1}(\mathbb{R})$ is actually a $K(\pi,1)$-space for the pure cactus group $PJ_n$, the orbifold homology of the quotient orbifold $[\overline{M}_0^{n+1}(\mathbb{R})/S_n]$ coincides with the group homology of $J_n$. Moreover, since $S_n$ is finite, the rational orbifold homology agrees with the rational homology of the coarse quotient. Consequently, the rational homology of $J_n$ is isomorphic to the rational homology of the quotient space, that is,

$$H_*(J_n;\mathbb{Q}) \cong H_*^{\mathrm{orb}}\left([\overline{M}_0^{n+1}(\mathbb{R})/S_n];\mathbb{Q}\right) \cong H_*\left(\overline{M}_0^{n+1}(\mathbb{R})/S_n;\mathbb{Q}\right).$$

Although the rational cohomology rings of the pure cactus groups have been determined [8], little is known about the homology of cactus groups. The aim of this paper is to compute the second homology of cactus groups. Our approach is inspired by [1], where Hopf's formula was applied to Coxeter groups and Artin groups to obtain the second mod 2 homology of all Artin groups.

*Notations.* For a group $G$, the abelianization of $G$ is denoted by $G^{\mathrm{ab}}$. Moreover, we write $H_n(G)$ for the $n$-th homology group of $G$ with trivial integral coefficients.

# 2 Preliminaries

## 2.1 Hopf's formula

We recall Hopf's formula, which gives a description of the second integral homology of an arbitrary group.

**Theorem 2.1 (Brown [3, Section II.5])** *Let $G$ be a group with presentation $\langle X \mid R\rangle$. Then*

$$H_2(G) \cong \frac{N \cap [F,F]}{[N,F]},$$

*where $F = F(X)$ is the free group on $X$ and $N = N(R)$ is the normal closure of $R$ on $F$.*

Hopf's formula satisfies the following naturality property.

**Lemma 2.2 (Brown [3, Section II.6, Exercise 3(b)])** *Let $G = F/N = \langle X \mid R\rangle$ and $G' = F'/N' = \langle X' \mid R'\rangle$ as in Theorem 2.1. Suppose a homomorphism $\alpha : G \to G'$ lifts to $\widetilde{\alpha} : F \to F'$. Then the following diagram commutes,*

$$\begin{array}{ccc} H_2(G) & \xrightarrow{\cong} & (N \cap [F,F])/[N,F] \\ {\scriptstyle H_2(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ H_2(G') & \xrightarrow[\cong]{} & (N' \cap [F',F'])/[N',F'] \end{array}$$

*where $\alpha_*$ is induced by $\widetilde{\alpha}$.*

**Lemma 2.3 (Akita–Liu [1, Lemma 3.6, 3.7])** *Let $G = F/N = \langle X \mid R\rangle$ as in Theorem 2.1. Then $N/[N, F]$ is abelian and for any $n \in N$ and $f \in F$, we have*

$$fnf^{-1} \equiv n \mod [N, F].$$

Lemma 2.3 implies that any element of $N/[N, F]$ can be represented by an element of the form

$$\prod_{r\in R} r^{n(r)} \in N,$$

for some integers $n(r) \in \mathbb{Z}$. By Hopf's formula, those elements that also lie in $[F, F]$, namely

$$\prod_{r\in R} r^{n(r)} \in N \cap [F, F],$$

represent all elements of $H_2(G)$. Therefore, we can easily obtain an upper bound for the number of generators of $H_2(G)$.

**Lemma 2.4 (Akita–Liu [1, Lemma 3.8])** *Let $G = F/N = \langle X \mid R\rangle$ as in Theorem 2.1. If $x, y, z \in F$ satisfy $[x, y], [x, z] \in N \cap [F, F]$, then*

$$[x, yz] \equiv [x, y][x, z] \mod [N, F].$$

## 2.2 First integral homology of cactus groups

Let $n \geq 2$ be an integer. The first integral homology group of $J_n$, which is isomorphic to the abelianization $J_n^{\mathrm{ab}}$, is an abelian group generated by the elements $\sigma_{p,q}$ with $1 \leq p < q \leq n$, subject to the following two relations:

- $\sigma_{p,q}^2 = 1 \quad \text{for} \quad 1 \leq p < q \leq n,$
- $\sigma_{p,q}\sigma_{r,s} = \sigma_{p+q-s,p+q-r}\sigma_{p,q} \quad \text{for} \quad [r, s] \subset [p, q].$

Since all generators commute in $H_1(J_n)$, the second relation implies that

$$\sigma_{r,s} = \sigma_{p+q-s,\, p+q-r} \quad \text{for} \quad [r, s] \subset [p, q],$$

which is equivalent to

$$\sigma_{1,t} = \sigma_{k,\, k+t-1} \quad \text{for} \quad 2 \leq t \leq n,\ 2 \leq k,\ k + t - 1 \leq n.$$

If we define $\sigma_t := \sigma_{1,t}$ for $2 \leq t \leq n$, then $H_1(J_n)$ is an abelian group generated by $\sigma_2, \dots, \sigma_n$, subject to the relations

$$\sigma_t^2 = 1 \quad \text{for} \quad 2 \leq t \leq n.$$

Consequently, we obtain the following two lemmas.

**Lemma 2.5** *For $n \geq 2$, the first integral homology of $J_n$ is an elementary abelian 2-group of rank $n - 1$ with basis $\{\sigma_t \mid 2 \leq t \leq n\}$.*

**Lemma 2.6** *For $n \geq 3$, the second integral homology of $J_n^{\mathrm{ab}}$ is an elementary abelian 2-group of rank $\binom{n-1}{2}$ with basis $\{[\sigma_i, \sigma_j] \mid 2 \leq i < j \leq n\}$.*

### 2.3 Minimal presentation of cactus groups

Chemin and Nanda obtained a minimal presentation for cactus groups in terms of generators and non-redundant relations as follows:

**Theorem 2.7 (Chemin–Nanda [4, Theorem A])** *For $n \geq 2$, the cactus group $J_n$ admits a presentation with generators $\sigma_i := \sigma_{1,i}$ for $2 \leq i \leq n$, subject to the following relations:*

*(i)* $\sigma_i^2 = 1$ *for* $2 \leq i \leq n$,

*(ii)* $(\sigma_k \sigma_i \sigma_k \sigma_j)^2 = 1$ *for* $4 \leq i + j \leq k \leq n, 2 \leq i \leq j$,

*(iii)* $\sigma_k \sigma_{i+j} \sigma_j \sigma_{i+j} = \sigma_{k-i} \sigma_j \sigma_{k-i} \sigma_k$ *for* $3 \leq i + j < k \leq n, i \geq 1, j \geq 2, i + j \leq k - i$.

*Moreover, this presentation is minimal in terms of the number of generators.*

We further simplify this minimal presentation. By straightforward calculations, the relations (b) and (c) in the following corollary are equivalent to relations (ii) and (iii) in Theorem 2.7, respectively.

**Corollary 2.8** *For $n \geq 2$, the cactus group $J_n$ admits a presentation with generators $\sigma_i := \sigma_{1,i}$ for $2 \leq i \leq n$ and the following relations:*

*(a)* $\sigma_i^2 = 1$ *for* $2 \leq i \leq n$,

*(b)* $[\sigma_k \sigma_i \sigma_k, \sigma_j] = 1$ *for* $4 \leq i + j \leq k \leq n, 2 \leq i \leq j$,

*(c)* $[\sigma_{k-i} \sigma_k \sigma_{i+j}, \sigma_j] = 1$ *for* $3 \leq i + j < k \leq n, i \geq 1, j \geq 2, i + j \leq k - i$.

*Moreover, this presentation is minimal in terms of the number of generators.*

We will use the minimal presentation in Corollary 2.8 to compute the second integral homology of cactus groups.

## 3 Main Results

Our main result is stated as follows.

**Theorem 3.1** *The second integral homology of $J_2$ and $J_3$ are trivial. For $n \geq 4$, the second integral homology of $J_n$ is a nontrivial elementary abelian 2-group of finite rank.*

As an immediate consequence of Theorem 3.1, we obtain the following corollary.

**Corollary 3.2** *For $n \geq 2$, the second rational homology of $J_n$ is trivial.*

## References


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